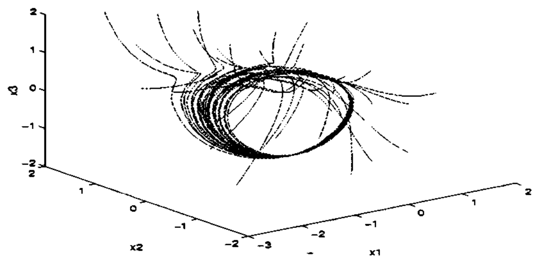

Fig. 4. Network with $g = \max(0, s)$.

can be observed in all of our simulation results. This further confirmed that the network processes the property of absolute periodicity.

V. Conclusions

In this brief, we have studied the absolute periodicity and absolute stability of delayed neural networks. We have obtained some simple and checkable conditions on networks' connection matrix to guarantee the absolute periodicity and absolute stability. The fact that neural networks have the property of absolute periodicity is an interesting dynamic behavior and we believe it will find more applications in practice.

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Fast Terminal Sliding-Mode Control Design for Nonlinear Dynamical Systems

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***Abstract*—In this brief, a fast terminal dynamics is proposed and used in the design of the sliding-mode control for single-input single-output nonlinear dynamical systems. The inherent dynamic properties of the fast terminal sliding modes are explored and conditions to ensure its applicability for control designs are obtained.**

I. Introduction

Sliding-mode control systems have been studied extensively and used in many applications [1] due to their robustness and simplicity. The sliding mode is attained by designing the control laws which drive the system to reach and remain on the intersection of a set of prescribed switching manifolds which are commonly selected as asymptotical stable linear switching hyperplanes. However, for high-precision control, the asymptotical stability may not deliver a fast convergence without imposing strong control force. Nonlinear switching manifolds such as the terminal sliding modes (TSMs), can

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improve the transient performance substantially. TSM control has been used successfully in control designs [2], [4], [5]. However, in comparison with the sliding-mode control based on linear switching hyperplanes, the existing TSM control may not deliver the same convergence performance when the system state is far away from the equilibrium, albeit its finite time convergence lies in its exponentially growing convergence rate when the state is near the equilibrium.

In this brief, we propose a new fast terminal sliding mode (FTSM) model that is able to combine the advantages of the TSM control and the conventional sliding-mode control (linear hyperplane based) together so that fast (finite time) transient convergence both at a distance from and at a close range of the equilibrium can be obtained. This model can deliver a control performance that cannot be realized by using either approach alone. We will fully explore the dynamic properties of the FTSMs and develop the control design based on FTSMs.

II. The Fast Terminal Sliding-Mode (FTSM) Concept

The TSM concept first introduced in [2] can be described as

$$s = \dot{x}_1 + \beta\, x_1^{q/p} = 0 \tag{1}$$

where $x_1 \in R^1$ is a scalar variable, and $\beta > 0$ and $p, q\ (p > q)$ are positive integers. Note that the parameters p and q must be odd integers and only the real solution is considered so that, for any real number x_1, $x_1^{q/p}$ is always a real number. It can be easily verified that, given any initial state $x_1(0) \neq 0$, the dynamics (1) will reach $x_1 = 0$ in a finite time determined by $t^s = (p/\beta(p-q))|x_1(0)|^{(p-q)/p}$. The equilibrium 0 is a terminal attractor [3], i.e., the state $x_1 = 0$ can be reached in a finite time and it is stable. The term "terminal" is referred to the equilibrium which can be reached in finite time and is stable. The reaching time t^s can be tuned by setting parameters $p,\ q,\ \beta$.

The introduction of the nonlinearity term $x_1^{q/p}$ improves the convergence toward equilibrium. The closer to equilibrium, the faster the convergence rate, resulting in finite time convergence. Note that although the terminal dynamics is not Liptchitz, for any nonzero initial condition, the solution in the forward time direction is unique [11].

It is should be noted that there is a close relationship between the TSM (1) and the time optimal control. In fact, the time optimal control for the double integrator system [7] can be approximated by a TSM model.

When the system state is far away from the equilibrium, the TSM model (1) does not prevail over the linear counterpart (setting $p = q$) since the term $x_1^{q/p}$ tends to reduce the magnitude of the convergence rate at a distance from equilibrium. One immediate solution is to introduce the following so-called fast terminal sliding-mode model:

$$s = \dot{x}_1 + \alpha\, x_1 + \beta\, x_1^{q/p} = 0 \tag{2}$$

where $\alpha,\ \beta > 0$. By doing so, we have $\dot{x}_1 = -\alpha\, x_1 - \beta\, x_1^{q/p}$. For properly chosen $q,\ p$, given an initial state $x_1(0) \neq 0$, the dynamics (2) will reach $x_1 = 0$ in finite time. The physical interpretation is: when x_1 is far away from zero, the approximate dynamics becomes $\dot{x}_1 = -\alpha x_1$ whose fast convergence when far away from zero is well understood. When close to $x_1 = 0$, the approximate dynamics becomes $\dot{x}_1 = -\beta x_1^{q/p}$ which is a terminal attractor [3]. More precisely, we can solve the differential (2) analytically. The exact time to reach zero, t^s, is determined by

$$t^s = \frac{p}{\alpha(p-q)}\left(\ln(\alpha x_1(0)^{(p-q)/p} + \beta) - \ln\beta\right) \tag{3}$$

and the equilibrium 0 is a terminal attractor.

The fast convergence performance of FTSM in comparison with the conventional linear hyperplane based sliding mode can be demonstrated by the following example. Consider $\alpha = 1$ and $\beta = 1$ and initial condition $x_1(0) = 1$. First let us assume $p = 3$ amd $q = 1$. From (3), one can easily find that the time to reach zero is $t^s = 1.039\,720\,770\,839\,92$. We now compare the above with the situation where p and q are set to 1. The simulation suggests that at approximate $t^s = 1.039\,699\,999\,999\,90$, for the case of $p = 3$ and $q = 1$, $x_1(t^s) = 0.000\,000\,091\,785\,40$ and for the case of $p = 1$ and $q = 1$, $x_1(t^s) = 0.125\,005\,192\,817\,75$. It is evident that the convergence rate of the FTSM is far better than its linear counterpart. The obvious reason is when close to the equilibrium, the convergence rate of the linear hyperplane based sliding mode remains a constant while the convergence rate of the FTSM grows exponentially.

Based on the the FTSM in (2), we now propose the following new recursive procedure for FTSM control of higher order systems:

$$s_1 = \dot{s}_0 + \alpha_0 s_0 + \beta_0 s_0^{q_0/p_0} \tag{4}$$

$$s_2 = \dot{s}_1 + \alpha_1 s_1 + \beta_1 s_1^{q_1/p_1} \tag{5}$$

$$\vdots \quad \vdots$$

$$s_{n-1} = \dot{s}_{n-2} + \alpha_{n-2} s_{n-2} + \beta_{n-2} s_{n-2}^{q_{n-2}/p_{n-2}} \tag{6}$$

where $s_0 = x_1$, $\alpha_i > 0$, $\beta_i > 0$ and q_i, $p_i (i = 0, \ldots, n-2)$ are positive odd numbers. The same analog applies that once $s_{n-1} = 0$ is reached, s_{n-2} will reach zero in finite time, and so will $s_{n-3},\ \ldots, s_0$. It is easy to establish that the time to reach the equilibrium is

$$\begin{aligned} T = &\sum_{i=1}^{n} t_i = t_n \\ &+ \sum_{i=1}^{n-1} \frac{p_{i-1}}{\alpha_{i-1}(p_{i-1}-q_{i-1})} \\ &\times \left(\ln(\alpha_{i-1} s_{i-1}(t_i)^{(p_{i-1}-q_{i-1})/p_{i-1}+p_{i-1}}) - \ln\beta_{i-1}\right) \end{aligned} \tag{7}$$

where t_n is the time to reach the terminal sliding mode s_{n-1} while

$$\begin{aligned} t_i = &\frac{p_{i-1}}{\alpha_{i-1}(p_{i-1}-q_{i-1})} \\ &\times \left(\ln(\alpha_{i-1} s_{i-1}(t_i)^{(p_{i-1}-q_{i-1})/p_{i-1}p_{i-1}}) - \ln\beta_{i-1}\right) \end{aligned}$$

for $i = n-1, \ldots, 1$ is the time from $s_i(t_i) \neq 0$ to $s_i(t_i + t_{i-1}) = 0$.

Note that the procedure (4)–(6) actually defines a path for the state x to converge to equilibrium. Indeed, if $s_{n-1} = 0$ is considered as $n-1$-dimensional flow in the state space, s_{n-2} can be considered as a subspace with dimension shrunk by unity. Therefore, $s_0 = 0$ will be the result of the $n-1$-dimensional dynamic space shrunk by $n-1$ times.

III. Control Design Using FTSMs

We consider the general nonlinear smooth single-input single-output (SISO) system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{8}$$

where f and g are the smooth vector fields on $\mathcal{R}^n$, h is the scalar smooth field on $\mathcal{R}^n$, and $u \in \mathcal{R}^1$.

Before we proceed, we introduce several notations. The Lie derivative of h with respect to f is defined as the directional derivative $\mathcal{L}_f h = \bigtriangledown h$ where $\bigtriangledown h = (\partial h/\partial x)$ representing the gradient of h. Higher order Lie derivatives can be defined recursively as $\mathcal{L}_f^i h = \bigtriangledown(\mathcal{L}_f^{i-1} h) f$

for $i = 1, 2, \ldots$ with $\mathcal{L}_f^0 h = h$ (see [8]). Without loss of generality, we assume that the SISO system (8) has relative degree n in a region $\Omega \in \mathcal{R}^n$.

With the assumption, in Ω, the nonlinear system (8) can be transformed to the following normal form:

$$\begin{aligned} \dot{z}_i &= z_{i+1} \quad i = 1, \ldots, n-1 \\ \dot{z}_n &= a(z) + b(z)u \end{aligned} \tag{9}$$

where $z_i = \mathcal{L}_f^{i-1} h(x)$ for $i = 1, \ldots, n$, $a(z) = \mathcal{L}_f^n h(x)$, and $b(z) = \mathcal{L}_g \mathcal{L}_f^{n-1} h(x) \neq 0$. The transformation matrix is easily seen as $z = \Phi(x) = [h(x), \mathcal{L}_f h(x), \ldots, \mathcal{L}_f^{n-1} h(x)]^T$. The dynamics (9) can also be written as

$$\dot{z} = A(z) + B(z)u \tag{10}$$

where $A(z) = [z_2, \ldots, z_{n-1}, a(z)]^T$ and $B(z) = [0, \ldots, 0, b(z)]^T$.

The sliding-mode control u should be designed so that $s_{n-1}\dot{s}_{n-1} < -K|s_{n-1}|, \quad K > 0$ therefore $s_{n-1} = 0$ can be reached in finite time. Hence, according to (4)–(6), $x_1 = 0$ is reached in finite time. Following are some results on the FTSM control of the SISO nonlinear systems.

Theorem 1: For the system (8), if the control u is designed as

$$u = u_{eq} + u_d \tag{11}$$

where

$$\begin{aligned} u_{eq} =& -b^{-1}(z) \\ & \cdot \left(a(z) + \sum_{k=0}^{n-2} \left(\alpha_k \mathcal{L}_{A+Bu}^{n-k-1} s_k + \beta_k \mathcal{L}_{A+Bu}^{n-k-1} s_k^{q_k/p_k} \right) \right), \\ u_d =& -b^{-1}(z) K \text{ sign } s_{n-1} \\ K >& 0 \end{aligned}$$

with $K > 0$ being a constant, then the system will reach the sliding mode $s_{n-1} = 0$ in finite time.

Proof: To ensure the finite time reachability of the sliding mode $s_{n-1} = 0$, the condition $s_{n-1}\,\dot{s}_{n-1} < -K|s_{n-1}|$ should be satisfied. Taking the first-order derivative of s_{n-1}, one has

$$\dot{s}_{n-1} = \ddot{s}_{n-2} + \alpha_{n-2}\dot{s}_{n-2} + \beta_{n-2}\mathcal{L}_{A+Bu} s_{n-2}^{q_{n-2}/p_{n-2}} \tag{12}$$

since $s_i = \dot{s}_{i-1} + \alpha_{i-1}s_{i-1} + \beta_{i-1}s_{i-1}^{q_{i-1}/p_{i-1}}$, for $i = n-1, n-2, \ldots, 1$ and the lth-order derivative of s_i is

$$\begin{aligned} \mathcal{L}_{A+Bu}^l s_i = \mathcal{L}_{A+Bu}^{l+1} \dot{s}_{i-1} + \alpha_{i-1}\mathcal{L}_{A+Bu}^l s_{i-1} \\ + \beta_{i-1}\mathcal{L}_{A+Bu}^l (s_{i-1}^{q_{i-1}/p_{i-1}}) \end{aligned}$$

then it can be easily calculated that

$$\begin{aligned} \dot{s}_{n-1} &= \mathcal{L}_{A+Bu}^n s_0 + \sum_{k=0}^{n-2} \beta_k \mathcal{L}_{A+Bu}^{n-k-1} s_k + \sum_{k=0}^{n-2} \beta_k \mathcal{L}_{A+Bu}^{n-k-1} s_k^{q_k/p_k} \\ &= \dot{z}_n + \sum_{k=0}^{n-2} \alpha_k \mathcal{L}_{A+Bu}^{n-k-1} s_k + \sum_{k=0}^{n-2} \beta_k \mathcal{L}_{A+Bu}^{n-k-1} s_k^{q_k/p_k}. \end{aligned} \tag{13}$$

Substituting the control (11) into (13) yields $s_{n-1}\dot{s}_{n-1} = -K|s_{n-1}|$ which means that the sliding mode $s_{n-1} = 0$ will then be reached in finite time. In fact, following Section II, from any initial state at $t_0 = 0$, the time to reach zero is $t_n = |S_{n-1}(0)|/k$. Hence, the proof of the theorem is completed. Q.E.D.

The control (11) involves calculation of the terms $\mathcal{L}_{A+Bu}^{n-k-1} s_k^{q_{k+1}/p_{k+1}}$, $k = 0, \ldots, n-2$ that is lengthy and trivial. Here we present a qualitative result for the calculation of these terms and also show that these terms are independent of the control u.

Theorem 2: For any $k \in \{0, 1, \ldots, n-2\}$,

$$\mathcal{L}_{A+Bu}^{n-k-1} s_k^{q_k/p_k} = f_k(z) \tag{14}$$

where f_k is a continuous nonlinear function.

Proof: We prove this proposition using the mathematical induction approach. Let $k = 0$, then apparently $\mathcal{L}_{A+Bu}^{n-1} s_0^{q_0/p_0} = \mathcal{L}_{A+Bu}^{n-1} z_1^{q_0/p_0} = f_0(z)$. Let $k = 1$, then $\mathcal{L}_{A+Bu}^{n-2} s_1^{q_1/p_1} = \mathcal{L}_{A+Bu}^{n-2}(z_2 + \alpha_0 z_1 + \beta_0 z_1^{q_0/p_0})^{q_1/p_1}$ which can be apparently expressed as $f_1(z)$. Assume for $k = k_0$, $\mathcal{L}_{A+Bu}^{n-k_0-1} s_{k_0}^{q_{k_0}/p_{k_0}} = f_{k_0}(z)$. Let us examine the case of $k = k_0 + 1$. Since

$$\begin{aligned} s_{k_0} &= f_{k_0}(\dot{s}_{k_0-1}, s_{k_0-1}) \\ &= f_{k_0}(\ddot{s}_{k_0-2}, \dot{s}_{k_0-2}, s_{k_0-2}) = \cdots \\ &= f_{k_0}(s_0^{(k_0)}, s_0^{(k_0-1)}, \ldots, s_0) \end{aligned}$$

and $s_0 = z_1$, then $s_{k_0}^{(l)}$ is a function of $z_{k_0+l+1}, z_{k_0+l}, \ldots, z_1$. Therefore, $\mathcal{L}_{A+Bu}^{n-k_0-2} s_{k_0+1}^{q_{k_0+1}/p_{k_0+1}}$ is a function of z. Q.E.D.

The parameters q_k, p_k must be chosen carefully in order to avoid the singularity because there are terms in $d^{n-k-1}/dt^{n-k-1} s_k^{q_k/p_k}$ which may contain nagative powers so that, when $s_{k-1} \to 0$, $u \to \infty$. This problem can be remedied by the following theorem.

Theorem 3: If

$$\frac{q_k}{p_k} > \frac{n-k-1}{n-k}$$

then, when $s_k \to 0$ sequentially from $k = n-2$ to $k = 0$, u is bounded.

Proof: From the rule for the nth derivative of a composite function [10], we have that for function $F(s)$

$$\begin{aligned} \frac{d^n}{dt^n} F(s) = \sum \frac{n!}{i_1! i_2! \ldots i_l!} \frac{d^m F}{ds^m} \left(\frac{\dot{s}}{1!}\right)^{i_1} \left(\frac{\ddot{s}}{2!}\right)^{i_2} \\ \times \left(\frac{s^{(3)}}{3!}\right)^{i_3} \cdots \left(\frac{s^{(l)}}{l!}\right)^{i_l} \end{aligned} \tag{15}$$

over all solutions in nonnegative integers of the equation $i_1 + 2i_2 + 3i_3 + \cdots l i_l = n$ and $m = i_1 + i_2 + i_3 + \cdots + i_l$. For simplicity, let $r = q_k/p_k$ and drop the index k. Since in the sliding mode $s_{k+1} = 0$,

$$s_{k+1} = \dot{s}_k + \alpha_k s_k + \beta_k s_k^{q_k/p_k} = \dot{s} + \alpha s + \beta s^r = 0$$

then $\dot{s} = \mathcal{O}(s^r)$ when $s \to 0$, where $\mathcal{O}$ is a complexity function. We therefore have $d^m F/ds^m = \mathcal{O}(s^{r-m})$ and $s^{(d)} = (\dot{s})^{(d-1)} = \mathcal{O}(s^{dr-(d-1)})$. So, we then have (16), shown at the bottom of the page. Hence when $s \to 0$, i.e., $s_k \to 0$, $(n+1)r - n = (n+1)q_k/p_k - n > 0$

$$\begin{aligned} \frac{d^n}{dt^n} F(s) &= \sum \mathcal{O}(s^{r-m}) \mathcal{O}(s^{i_1 r}) \mathcal{O}\left(s^{i_2(2r-1)}\right) \cdots \mathcal{O}\left(s^{i_l(lr-(l-1))}\right) \\ &= \sum \mathcal{O}(s^{r-m}) \mathcal{O}\left(s^{r(i_1+2i_2+3i_3+\cdots+li_l)+(i_1+i_2+i_3+\cdots+i_l)-(i_1+2i_2+3i_3+\cdots+li_l)}\right) \\ &= \sum \mathcal{O}(s^{r-m}) \mathcal{O}(s^{nr+m-n}) = \mathcal{O}\left(y^{(n+1)r-n}\right) \end{aligned} \tag{16}$$

will ensure that (16) is bounded. Also from the above analysis, we have $d^n/dt^n s = (\dot{s})^{(n-1)} = \mathcal{O}(s^r)^{(n-1)} = \mathcal{O}(s^{nr-n+1})$. Hence when $s \to 0$, $nr - n + 1 = nq_k/p_k - n + 1 > 0$ will ensure that $d^n/dt^n s$ is bounded. With the above expressions in mind, the control (11) can be rewritten as

$$\begin{aligned} u = &- b^{-1}(z) \\ &\cdot \Bigg(a(z) + \sum_{k=0}^{n-2} \bigg(\mathcal{O}\left(s_k^{(n-k-1)q_k/p_k-(n-k-1)+1} \right) \\ &+ \mathcal{O}\left(s_k^{(n-k)q_k/p_k-(n-k-1)} \right) \bigg) \\ &+ K \text{ sgn } s_{n-1} \Bigg). \end{aligned} \tag{17}$$

For the second term of (17) to be bounded while $s_k \to 0$, it is sufficient that $(n-k)q_k/p_k - (n-k-1) > 0$ for $s_k \to 0$ sequentially from $k = n-2$ to $k = 0$ so that the control u is bounded. Q.E.D.

Another kind of singularity is that, during the transient process toward the FTSMs, the control may become singular if some s_i becomes zero. This problem can be avoided by prohibiting the trajectory from reaching any of the switching manifolds $s_i = 0$ $(i = 0, 1, \ldots, h-1)$ before $s_h = 0$ is reached $(0 < h \le n-1)$. One can use the two-phase control strategy in [6] to avoid this. Once $s_{n-1} = 0$ is reached, the singularity problem will no longer exist and then trajectory $z(t)$ will first reach $s_{n-1} = 0$ and then $s_{n-2} = 0, \ldots, s_0 = 0$ sequentially.

For the nonlinear system (8) with external disturbances and noises v, $\dot{x} = f(x) + g(x)u + d(x)v$, $y = h(x)$, if $d(x) \in \text{range}(g(x))$, i.e., the matching condition is satisfied [9], then a slight modification of control (11) will guarantee the stability, the reachability of the sliding modes, and finite time reachability of the system equilibrium. Our recent study on robot control has shown that FTSM control has a better control precision and robustness than its linear counterpart (the results will be reported elsewhere). In fact, FTSM control is a high gain control when near the equilibrium.

It should be noted that many problems, such as chaos synchronization, chaotic communication, filter design, and others in the area of circuits and systems, can be formulated in the control problem framework. Making use of the principle developed in this paper will result in a high-precision control/tracking/estimation performance.

IV. CONCLUSION

We have proposed a fast terminal dynamics is for the design of sliding-mode control for SISO nonlinear dynamical systems. The inherent dynamic properties of the FTSMs have been explored and conditions to ensure its applicability for control design have been derived. Simulation results of a chaos synchronization problem and control of a flexible joint robot arm have shown the effectiveness of the control strategy developed. The FTSM control has demonstrated its superior performance. It should be noted the control strategy developed in this paper can be applied to a wide range of applications in circuits and systems, for example, in chaos communication and synchronization, filter designs, etc. Future work will be conducted in applying this mechanism to more general problems in control, optimization, and identification.

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